Shadows of rotating black holes approximated by Dürer-Pascal limaçons

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Abstract

It is demonstrated that the shadows of rotating charged black holes are well approximated by Dürer-Pascal limaçons. This is done by comparing plots of the shadows as derived by closed photon orbits [4] to the limaçons.

Contents

1 Introduction

Although a black hole is not visible, it may be observable nonetheless — it casts a shadow if it is in front of a bright background. In the early 1970's Bardeen [1] was the first to study the apparent shape of an extremely rotating black hole, later Luminet [12] visualized a Schwarzschild black hole with an accretion disc around it. In the 1990's Quien, Wehrse and Kindl [14] plotted accretion discs around an extremely rotating black hole as viewed from different angles of latitude. Supplementing these numerical approaches, the author [4]

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studied analytically the closed photon orbits in general Kerr-Newman space-times, even in cases where the so-called cosmic censorship is violated.

The observability of black hole shadows in the near future is very realistic. Recently, great interest emerged especially for the observability of the black hole in the center of our Milky Way, Sgr A* [2, 5, 8, 10].

The present article is focussed on an introduction of the optics of black hole and a comfortable representation of shadow shapes. It results in the discovery that the shadows are well approximated by Dürer-Pascal limaçons. These plane algebraic curves are an interesting issue on their own, being first mentioned by Albrecht Dürer and later reinvented by Étienne Pascal.

The article proceeds as follows. In section 3 a concise overview of the basic principles and the theory concerning shadows of charged rotating black holes is given. Section 4 yields a short introduction to the mathematics of limaçon curves, before limaçons as approximations of rotating black hole shadows are studied in section 5. A short discussion concludes the paper.

2 Motivation

What is a black hole? It is commonly known that a black hole is a mass being concentrated to such a small space region that its space-time curvature, or its gravity, is strong enough to even capture light, not to mention massive particles. Although German astronomer Karl Schwarzschild discovered this space-time as an exact solution of the field equations of Einstein's —then only a few months old— general relativity, their physical relevance was only recognized decades later. It was John A. Wheeler, a student of Albert Einstein, who introduced the name "black hole".

In 1963 Roy Kerr discovered the solution of a rotating mass, which is considerably more complicated, and it could be generalized to a rotating charged solution a few years later by Ted Newman and his coworkers. This solution represents, besides even more general mass fields, especially the most general black hole. It was a long and hard mathematical work to prove the so-called uniqueness theorem of the Kerr-Newman solution, done by several theoretical physicists from 1967 to 1982 (Israel, Carter, Hawking, Robinson, Mazur). It states that the only stationary and 'asymptotically flat' solution of the Einstein equations is the Kerr-Newman solution (see [9] for details and further references). In particular, this means that *any* black hole is characterized *only by the three parameters* mass, rotation, and electrical charge — regardless of the original structure of the matter having been 'swallowed' by the black hole.

This remarkable property is often referred to as the 'no-hair' theorem and gave rise to an important theoretical controversy whether the information about physical quantities of matter vanishing in the black hole is completely lost. If this was true, the energy conservation would be violated eventually and the universe would heat up. This is the 'information paradox', and the only known ansatz to solve it is by superstring theory providing a fundamental derivation of black hole entropy [11, 15, 16].

Another interesting consequence of the no-hair property is that the outline of the shadow casted by a black hole is determined only by its three parameters and the relative position of the observer. Each parameter constellation has its own characteristic shape, astronomers can observe the rotation and charge relative to the mass optically. A systematic study of black hole shadows [4] was first presented in 2000.

When I computed the apparent black hole shadow outlines, I was amazed at their similarity to limaçons, special algebraic curves. I tried to investigate whether there is a deeper relationship between them. However, I did not find one, unless the fact that the similarities are rather strong in a wide range of the parameters. In this paper I will shortly present this property and by the way introduce into the interesting subject of limaçons as well as into the optics of black holes.

3 The shadow of a black hole

To compute the shapes of black hole shadows, we first have to study light rays in general relativity. In the following section, we therefore consider geometrical optics in curved space-times.

3.1 Geometrical optics

The key idea to compute the apparent shape of a general black hole relies on geometrical optics, where an electromagnetic wave propagates approximately on a congruence of light rays perpendicular to the wave fronts. According to general relativity, light rays are bended lines in a space-time curved by existing masses.

However, the geometric optics approximation is valid only if the wavelength λ is small as compared to the typical radius of curvature in the considered space-time region. For a black hole the radius of curvature, R, is given by the reciprocal value of the typical component of the Riemann tensor,

$$R \approx \sqrt{\frac{r^3}{r_+}}, \tag{1}$$

where r is the distance from the black hole center, and r_+ is the event horizon of the black hole [3, 4].

E.g., for a black hole of solar mass the event radius r_+ is about 1.5 km, i.e., the radius of curvature $R \approx 1.5$–2 km in the close neighborhood of the event horizon. Therefore, for a black hole of solar mass and for a medium wave radiation (with λ between 100 m and 1 km) originated close to its event horizon geometrical optics is not an admissible approximation, but for very high frequency it does, a fortiori for even shorter waves like infrared waves, light, or x-rays.

For the center of the milky way galaxy, presumably a giant black hole of about 2.6 million solar masses, the radius of curvature is about 4 million km, i.e., the error of a measurement of a long wave source ($\lambda \approx 10\,\text{km}$) close to the event horizon is about 1:400 000.

If we put up with inaccuracy of geometric optics, we get aware of the world around us as an optical illusion. Since the light rays reaching an observer in infinity are curved, they appear to come from a totally different direction. To represent the apparent source

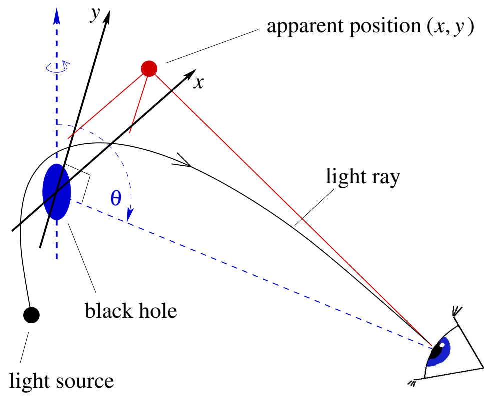

Figure 1: The (x,y) coordinates, indicating the apparent position of a light ray with respect to the observer's projection plane containing the center of the space-time: x denotes the apparent distance from the rotation axis, and y the projection of the rotation axis itself (dashed line). θ denotes the angle of latitude (reaching from the north pole at $\theta = 0$ to the south pole at $\theta = \pi$).

location with respect to the projection plane of the observer the (x,y) coordinates are used, as explained in Fig. 1.

3.2 Closed photon orbits

Further analysis of the light rays in a general Kerr-Newman space-time with a singularity shows the existence of photons which move on closed orbits [1, 3, 4]. They describe the limit of the innermost photon orbits coming from infinity and escaping back to infinity. (Here "infinity" means the asymptotically flat space-time region far away from the black hole; yet already at a distance of 25 times the event radius, $25r_+$, the radius of curvature according to (1) is $R \approx 125r_+$, so the asymptotically flat regions are reached rather quickly.)

Thus for an observer at infinity, who sees the black hole in front of an illuminated background in the asymptotic flat region, the black hole casts a shadow which in the approximation of geometrical optics is given by the set of its closed photon orbits.

According to [4] the nonlinear system of differential equations governing closed photon orbits can be reduced to a gradient system with a potential function being a polynomial $F(r)$ of degree 5 with respect to the radial photonic distance r. A necessary condition for a closed orbit with $r = r_c$, with r_c a constant, then is simply the vanishing of the first derivative of the potential function, $\partial F(r_c)/\partial r = 0$. Remarkably, in a Kerr-Newman space-time any closed photon orbit *implies* that the second derivative of the potential functions vanishes, $\partial F^2/\partial r^2 = 0$.[1] Therefore, any closed photon orbit in a Kerr-Newman space-time is unstable. Mathematically, such a closed orbit corresponds to exactly one point in the bifurcation set of the potential function F, which is the so-called swallow tail [4].

[1] This fact is mentioned in [1, 3], but it seems be proved presumably at first in [4, §3.1].

Let us consider a Kerr-Newman space-time with mass $\mathfrak{M}$ (in kg), angular momentum $\mathfrak{J}$ (in $\mathrm{kg\,m^2\,s^{-1}}$), and electrical charge $\mathfrak{q}$ (in $\mathrm{kg^{1/2}\,m^{3/2}\,s^{-1}}$). Then the related geometrized parameters M, a, and Q specifying the space-time are

$$M = \frac{G\mathfrak{M}}{c^2}, \quad a = \frac{\mathfrak{J}}{c\mathfrak{M}}, \quad Q = \frac{\sqrt{G}\,\mathfrak{q}}{c}. \tag{2}$$

Algebraically, a photon moving on a closed orbit with radius r in a Kerr-Newman space-time with $a \neq 0$ has the apparent position in the (x,y) reference frame of an asymptotic observer located in the angle of latitude θ (cf. Fig. 1) given by

$$x = \frac{r\Delta + rQ^2 - M(r^2 - a^2)}{a(r-M)\sin\theta}, \tag{3}$$

$$y^2 = \frac{4r^2\Delta}{(r-M)^2} - (x + a\sin\theta)^2. \tag{4}$$

where $\Delta = r^2 - 2Mr + a^2 + Q^2$. Cf. [4, Eqs. (51, 65, 69)[2]]; for the Kerr case ($Q = 0$) see also [3, §63 Eqs. (224, 225, 241)]. For the non-rotating case $a = 0$, the (x,y)-position is determined by the circle equation

$$x^2 + y^2 = \frac{r^4}{\Delta}, \quad \text{with } r = \tfrac{3}{2}M\left(1 + \sqrt{1 - \frac{8Q^2}{9M^2}}\,\right). \tag{5}$$

As long as the parameters are chosen such that the space-time satisfies the “cosmic censorship hypothesis” [13], prohibiting naked singularities, viz. $M^2 \geqq a^2 + Q^2$ [6], these formulas yield well-defined closed curves on the celestial sphere of the asymptotic observer ([4, Fig. 2] or Fig. 5 below).

The variable x in Eq. (3), considered as a function of r, is monotone increasing with respect to r in the region $r > M$. It thus can be inverted. Since the nominator of the right hand side of Eq. (3) is a cubic polynomial in r, we achieve by the Cardano formula

$$r = M \;+\; \sqrt[3]{q + \sqrt{p^3 + q^2}} \;+\; \sqrt[3]{q - \sqrt{p^3 + q^2}} \tag{6}$$

where

$$q = M(M^2 - a^2 - Q^2), \qquad p = -\tfrac{1}{3}\,(a\,x\sin\theta + 3M^2 - a^2 - 2Q^2). \tag{7}$$

Note that q is a constant with respect to x, and $p = p(x)$.

4 Dürer-Pascal limaçon

The Dürer-Pascal limaçon, or shortly Pascal’s limaçon, is a plane curve generated by the locus of points which in polar coordinates (r,φ) satisfy the equation

$$r = A\cos\varphi + B, \tag{8}$$

[2] with x replaced by α, and y by β.

with $0 \leqq \varphi < 2\pi$, and A, $B \geqq 0$ (Fig. 2). Geometrically, a point of the limaçon is on a line at a fixed distance B from the point of intersection of the line with a fixed circle of radius $A/2$, as the line revolves about a point on the circumference of the circle. Since the equation of

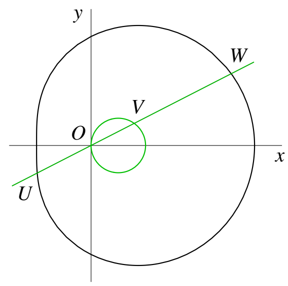

Figure 2: Dürer-Pascal limaçon for $2A \leqq B$. The radius of the auxiliary circle (light line) is $A/2$, and the identities $\overline{VW} = \overline{UV} = B$ hold.

the circle is given by $(x - \frac{A}{2})^2 + y^2 = \frac{A^2}{4}$, i.e. in polar coordinates, $r = A\cos\varphi$, the limaçon is also called the conchoid of the circle.

Depending on the values of A and B, the form of the limaçon differs (Fig. 3). For $2A \leqq B$

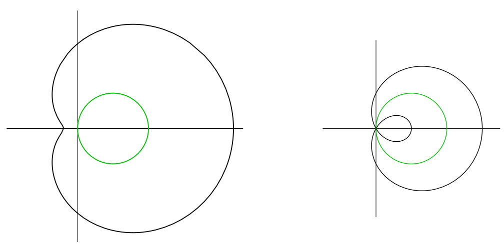

Figure 3: Dürer-Pascal limaçon for $A < B < 2A$ (left hand side) and for $A > B$ (right hand side).

it is a convex curve, for $B = A$ it is a cardioid, and for $A > B > 0$ it has a singular point where it intersects itself. Some special cases are listed in the table:

Condition	**Limaçon**
$A = 0 < B$	circle of radius B
$A = B > 0$	cardioid
$A > 0 = B$	circle of radius $A/2$

The area $\mathscr{A}$ of a limaçon with parameters A and B is given by

$$\mathscr{A} = \frac{\pi}{2}\left(A^2 + 2B^2\right). \tag{9}$$

(For the case $A > B$, the inner loop then is computed double.) Written in Cartesian coordinates (x,y), the limaçon equation (8) reads

$$(x^2 + y^2 - Ax)^2 - B^2(x^2 + y^2) = 0. \tag{10}$$

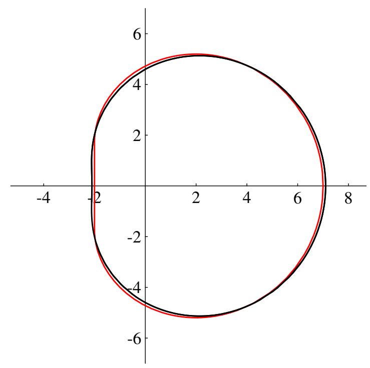

Figure 4: The limaçon (black curve) with the parameters $A = 2.50$ and $B = 4.6$, in comparison to the shadow of a black hole with (extreme) rotation parameter $a = M$ (light curve) in the reference frame (x, y) of an asymptotic observer in the equatorial plane $\theta = \frac{\pi}{2}$, according to Fig. 1. The unit length is the (geometrized) black hole mass M.

Thus the limaçon is an algebraic curve of 4th order with genus $g = 2$ for $2A > B$, or genus $g = 3$ for the convex case $2A \leqq B$.

The limaçon is named after Étienne Pascal (1588–1651), father of Blaise Pascal, though Albrecht Dürer (1471–1528) had already published the curve in 1525 [7, Fig. 40]; he called it *Spinnenlinie* (“spider curve”), due to his special manner of construction [L2].

5 Black hole shadows and limaçons

Regarding the shadow forms of rotating black holes, the border of which is determined by equations (3) and (4), one finds a striking similarity with Dürer-Pascal limaçons. Although a closer analysis shows that the shadows are *not* exact limaçons, it is worthwile to consider which limaçon parameters fit to the shadows of some black hole with given rotation parameter a, as is listed the table:

a	.0	.1	.2	.3	.4	.5	.6	.7	.8	.866	.9	.95	.98	.99	.999	1.0
A	.0	.21	.40	.61	.82	1.01	1.24	1.48	1.71	1.90	2.01	2.17	2.30	2.38	2.47	2.50
B	$\sqrt{27}$	5.19	5.18	5.16	5.14	5.11	5.07	5.02	4.93	4.86	4.82	4.74	4.68	4.65	4.62	4.60

(11)

Remarkably, the inaccuracy increases for increasing rotation. For illustrative purposes, the case of an extremely rotating (uncharged) black hole with $a = M$ (and $Q = 0$) is shown in Fig. 4. The differences between the shadow border curve and the limaçon are distinct, though they are relatively small. Since the shadow curve itself, being derived from geometrical optics, is an approximation, it may also be considered as a limaçon.

Various black hole shadows, as seen by an observer in the equatorial plane far away from the black hole, are shown in Fig. 5. Here also shadows of electrically charged black holes are plotted. It is remarkable, that the shadows of any black hole observed in their equatorial planes is well approximated by a limaçon.

An observer of a black hole, however, is rarely located exactly in the equatorial plane of the black hole. Thus a general formula describing the shadow curve with respect to the

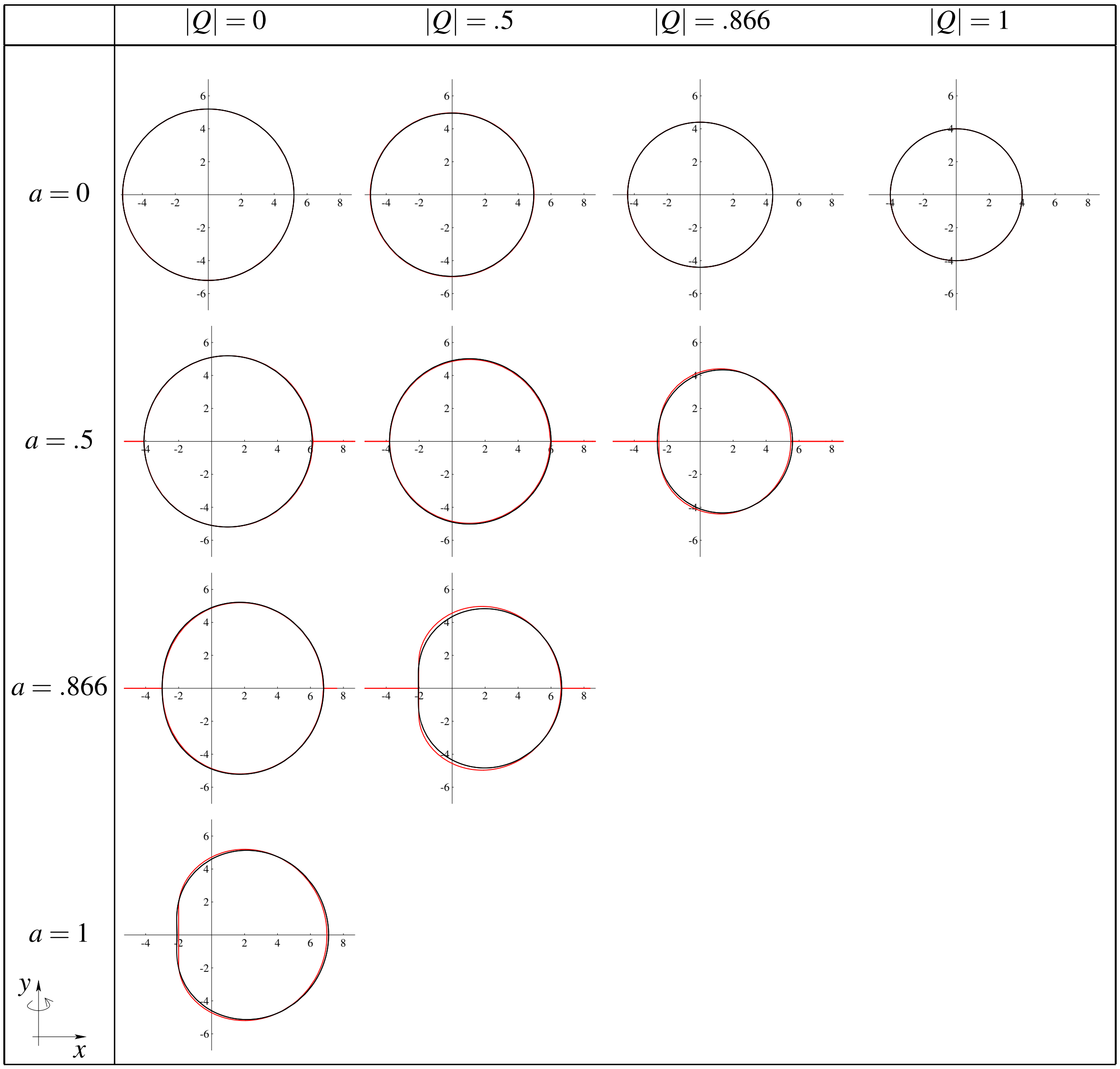

Figure 5: Limaçons (black lines) in comparison to black hole shadows (light lines) of various rotation a and electrical charge Q in the reference frame (x, y) of an asymptotic observer in the equatorial plane $\theta = \frac{\pi}{2}$, according to Fig. 1. The unit length is the (geometrized) mass M.

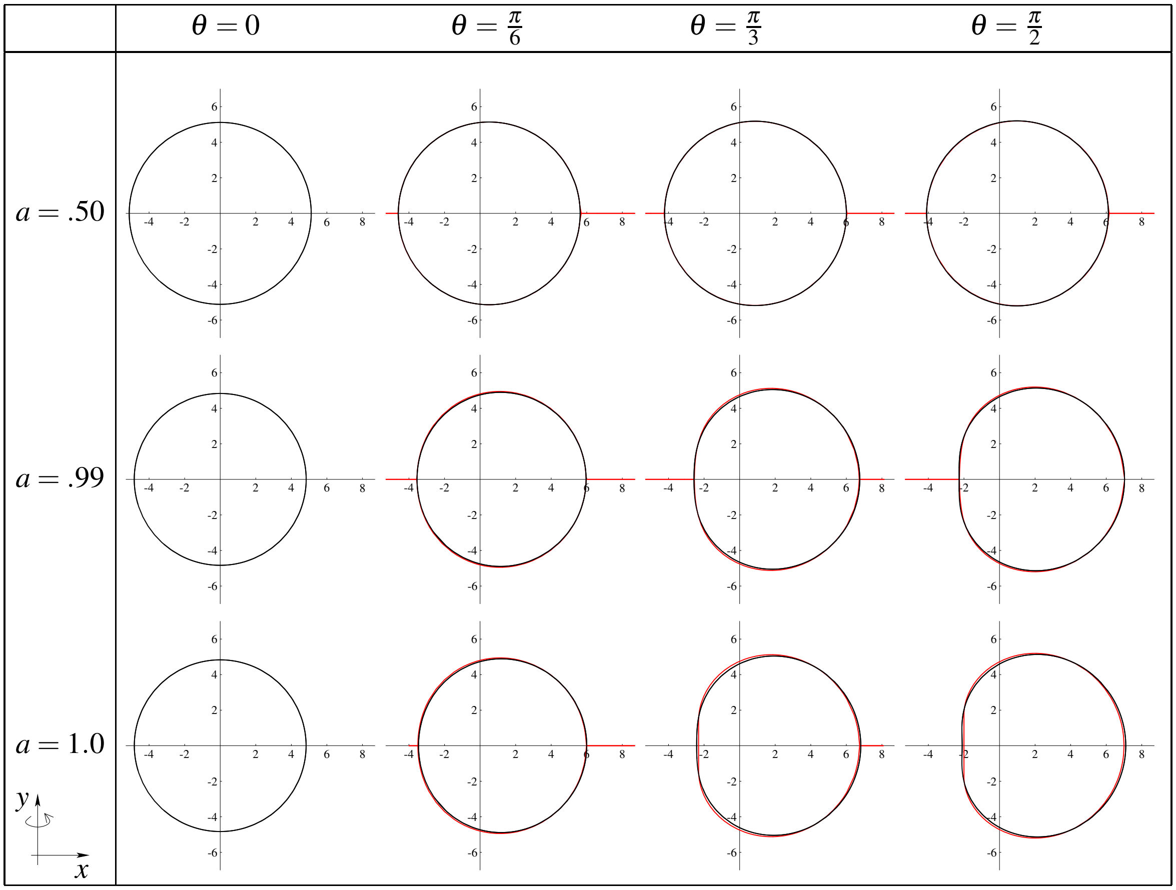

Figure 6: Quater-round trips around various uncharged black holes, starting at the north pole direction ($\theta = 0$) and ending in the equatorial plane ($\theta = \frac{\pi}{2}$) (the other quarters are symmetric to this). Shown are the limaçons (black lines) in comparison to outlines black hole shadows (light lines) of various rotation parameters a, again in the reference frame (x,y) of the asymptotic observer in the actual θ-plane, according to Fig. 1. The unit length is M.

latitude θ has to be given. Since the curve in the (x,y) reference frame of the observer determined by Equ. (3) and (4), due to the circular orbits of the space-time, are explicitly depending on the latitude θ —with the remarkable exception of the poles $\theta = 0$ and $\theta = \pi$— the idea is obvious to investigate whether the apparent shadow curves as seen at varying latitudes θ could also be approximated by limaçons. And in fact they can. A rough inspection for the uncharged cases $Q = 0$ shows that they are rather well approximated by a limaçon (8), where the paramters A and B are given by the following simple Fourier series,

$$A_a(\theta) = A_a \sin\theta + .2\, a \sin^3\theta \cos^2\theta, \tag{12}$$

$$B_a(\theta) = B_a + .23\, M \left(1 - \sqrt{1 - \frac{a^4}{M^4}}\right) \cos^4\theta, \tag{13}$$

with A_0 and B_0 being the linearly interpolations of the respective values in Table (11). In

Fig. 6 the various apparent shadows as viewed from different latitudes are plotted, for a moderately rotating black hole ($a = .50$) and two extremely rotating ones ($a = .99$ and $a = 1.0$). Each row in the figure can be imagined as the first quarter of a round trip around the black hole, starting at the north pole direction ($\theta = 0$) and ending at the equatorial plane ($\theta = \frac{\pi}{2}$).

6 Discussion

This article showed the approximation of rotating black hole shadows by Dürer-Pascal limaçons, showing plots of various shadows and their adapted limaçon. Moreover it analyzed the conditions under which black hole shadows are observable.

What are the potential advantages of limaçons as representations of black hole shadows? First, limaçons are much easier to describe than the complicated curves determined by Equations (3) and (4). Instead of these equations, for the chargeless cases $Q = 0$ only Table (11) and the latitude-depending relations (12) and (13) determining the limaçon parameters A and B has to be regarded. One benefit is that the limaçons describe the shadow for observers near the pole directions, $\theta = 0$ or $\theta = \pi$, contrary to (3) and (4).

Second, the computation of the area now is an easy task by Equation (9). This may be important when computing intensity or luminosity differences due to absorption of background radiation.

It is true, the black hole shadows definitely are no limaçons, except in the spherically symmetric Schwarzschild case, where it is a circle. Their similarities are remarkable, though. Future work should investigate the properties of the shadow outlines as plane algebraic curves. Perhaps this will give further insight into the symmetry-breaking effect of rotation in general relativity.

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Web Links

[L1] *http://www.math-it.org/Mathematik/Astronomie/schatten.html* — an interactive animation showing black hole shadows

[L2] *http://haegar.fh-swf.de/spielwiese/spinnenlinie.pdf* — the original limaçon construction of Albrecht Dürer